"The Longitudinal Stability of Aerial Gliders." By G. H. Bryan, Sc.D., F.R.S., and W. E. Williams, B.Sc., University College of North Wales.

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1. *Introduction.*

The main difficulty connected with the attempt to fly by means of a machine heavier than air is that of longitudinal stability. It is not difficult to construct an aeroplane system which shall be *transversely* stable.

But for this difficulty the problem of artificial flight would probably have been solved already. Experiments in gliding under gravity have been always made with machines not too large to be kept balanced by the skill of the experimenter, and the glides, though undoubtedly successful, have been of short duration. Experiments have invariably stopped short of the performance of continuous flight by a mechanically propelled machine.

The problem of artificial flight is hardly likely to be solved until the conditions of longitudinal stability of an aeroplane system have been reduced to a matter of pure mathematical calculation.

A theoretical investigation, even if calculated under conditions slightly different to those occurring in nature, will serve as a basis of comparison by which experimental results can be co-ordinated and interpreted in their true light.

The object of these investigations is (1) to show that the longitudinal stability of aeroplane systems *can* be made the subject of mathematical calculation; (2) to draw the attention of those interested in the problem of artificial flight to the necessity of acquiring further experimental knowledge concerning the quantities on which this stability is shown to depend.

2. *General Investigation of the Longitudinal Stability of any Symmetrical Aeroplane System.*

Consider any system of aeroplanes, having a plane of symmetry, descending in a vertical plane, in air or in any resisting medium whatever. To specify the motion (which we suppose to be two-dimensional) choose two axes at right angles fixed in the body, having the centre of gravity as origin.

The motion will be completely determined if at every instant we know—

(1) the angle θ which the axis of x makes with the vertical.*

(2) the velocity components u, v, of the body along the two axes.

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* This angle is supposed measured from the *downward drawn* vertical in the positive direction. The axis of y must be drawn *above* the horizontal at an inclination of θ.

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The angular velocity of rotation of the body will be $d\theta/dt$ or $\dot{\theta}$.

We shall use m to denote the mass of the body, mk^2 its moment of inertia about the centre of gravity.

Whatever be the law of resistance, the resistances of the air on the several parts of the system will in general be functions of u, v and $\dot{\theta}$. These resistances are always equivalent to two forces, which we shall call $m\mathrm{X}$ and $m\mathrm{Y}$, along the axes, and a couple $m\mathrm{G}$ about the origin, so that X, Y, G, denote the forces and couple, divided by the mass of the body.

The equations of motion of the body are—

$$\left.\begin{aligned} m\left(\frac{du}{dt}-v\frac{d\theta}{dt}\right) &= mg\cos\theta - m\mathrm{X}, \\ m\left(\frac{dv}{dt}+u\frac{d\theta}{dt}\right) &= -mg\sin\theta - m\mathrm{Y}, \\ mk^2\frac{d^2\theta}{dt^2} &= -m\mathrm{G}. \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (1).$$

In *steady motion* u, v and θ are constant, and equations (1) give

$$0 = g\cos\theta - \mathrm{X}, \quad 0 = -g\sin\theta - \mathrm{Y}, \quad 0 = -\mathrm{G} \quad \ldots\ldots \quad (2).$$

Knowing the forms of the aeroplanes and other parts of the system and the law of resistance, X, Y, G are known functions of u, v and $\dot{\theta}$. Moreover, in steady motion, $\dot{\theta} = 0$.

Equations (2) thus determine the values of u, v, θ, for steady motion.

Fluctuations about Steady Motion.—We must now examine what happens when the system is slightly disturbed from its state of steady motion. Let the disturbance be represented at time t by small increases δu, δv, $\delta\theta$ in the values of u, v, θ, and let u_0, v_0, θ_0 be their values in steady motion, so that in the disturbed motion,

$$u = u_0 + \delta u, \qquad v = v_0 + \delta v, \qquad \theta = \theta_0 + \delta\theta.$$

Also let X_0, Y_0, G_0 be the forces and couple in the steady state; then in the disturbed state we have, neglecting small quantities of the second order,

$$\mathrm{X} = \mathrm{X}_0 + \delta u\frac{d\mathrm{X}}{du} + \delta v\frac{d\mathrm{X}}{dv} + \delta\dot{\theta}\frac{d\mathrm{X}}{d\dot{\theta}} \quad \ldots\ldots\ldots\ldots\ldots \quad (3),$$

and two similar equations for Y and G.

We shall denote differential coefficients such as $d\mathrm{X}/du \ldots d\mathrm{G}/d\dot{\theta}$ by $\mathrm{X}_u \ldots \mathrm{G}_\theta$.

Substituting in the equations of motion, we obtain, to the first order,

$$\frac{d}{dt}\delta u - v_0\frac{d}{dt}\delta\theta = -g\sin\theta_0\delta\theta - \mathrm{X}_u\delta u - \mathrm{X}_v\delta v - \mathrm{X}_{\dot\theta}\delta\dot\theta,$$

$$\frac{d}{dt}\delta v + u_0\frac{d}{dt}\delta\theta = -g\cos\theta_0\delta\theta - \mathrm{Y}_u\delta u - \mathrm{Y}_v\delta v - \mathrm{Y}_{\dot\theta}\delta\dot\theta,$$

$$k^2\frac{d^2}{dt^2}(\delta\theta) = -\mathrm{G}_u\delta u - \mathrm{G}_v\delta v - \mathrm{G}_{\dot\theta}\delta\dot\theta.$$

To solve these equations, put

$$\delta u = \mathrm{P}e^{\lambda t}, \qquad \delta v = \mathrm{Q}\epsilon^{\lambda t}, \qquad \delta\theta = \mathrm{R}\epsilon^{\lambda t}.$$

Substituting, dividing by $\epsilon^{\lambda t}$, and re-arranging the terms, we have

$$\begin{aligned}
\mathrm{P}(\lambda+\mathrm{X}_u)+\mathrm{QX}_v+\mathrm{R}(\lambda\mathrm{X}_{\dot\theta} - \lambda v_0+g\sin\theta_0) &= 0,\\
\mathrm{PY}_u+\mathrm{Q}(\lambda+\mathrm{Y}_v)+\mathrm{R}(\lambda\mathrm{Y}_{\dot\theta} + \lambda u_0+g\cos\theta_0) &= 0,\\
\mathrm{PG}_u+\mathrm{QG}_v+\mathrm{R}(\lambda^2k^2+\lambda\mathrm{G}_{\dot\theta}) &= 0.
\end{aligned}$$

Eliminating P, Q, R,

$$\begin{vmatrix}
\lambda+\mathrm{X}_u, & \mathrm{X}_v, & -\lambda v_0+\lambda\mathrm{X}_{\dot\theta}+g\sin\theta_0\\
\mathrm{Y}_u, & \lambda+\mathrm{Y}_v, & \lambda u_0+\lambda\mathrm{Y}_{\dot\theta}+g\cos\theta_0\\
\mathrm{G}_u, & \mathrm{G}_v, & \lambda^2k^2+\lambda\mathrm{G}_{\dot\theta}
\end{vmatrix} = 0 \quad (4).$$

If the determinant be expanded in powers of λ we get an equation of the form

$$\mathrm{A}\lambda^4+\mathrm{B}\lambda^3+\mathrm{C}\lambda^2+\mathrm{D}\lambda+\mathrm{E} = 0 \quad\ldots\ldots\ldots\ldots\quad (4a),$$

where

$$\left.\begin{aligned}
\mathrm{A} &= k^2,\\
\mathrm{B} &= k^2(\mathrm{X}_u+\mathrm{Y}_v)+\mathrm{G}_{\dot\theta},\\
\mathrm{C} &= k^2(\mathrm{X}_u\mathrm{Y}_v-\mathrm{X}_v\mathrm{Y}_u)+v_0\mathrm{G}_u-u_0\mathrm{G}_v-\mathrm{X}_{\dot\theta}\mathrm{G}_u-\mathrm{Y}_{\dot\theta}\mathrm{G}_v,\\
\mathrm{D} &= u_0(\mathrm{X}_v\mathrm{G}_u-\mathrm{X}_u\mathrm{G}_v)+v_0(\mathrm{Y}_v\mathrm{G}_u-\mathrm{Y}_u\mathrm{G}_v),\\
&\quad -g\sin\theta_0\mathrm{G}_u-g\cos\theta_0\mathrm{G}_v+\mathrm{Y}_{\dot\theta}(\mathrm{X}_v\mathrm{G}_u-\mathrm{X}_u\mathrm{G}_v)\\
&\quad -\mathrm{X}_{\dot\theta}(\mathrm{Y}_v\mathrm{G}_u-\mathrm{Y}_u\mathrm{G}_v)+\mathrm{G}_{\dot\theta}(\mathrm{X}_u\mathrm{Y}_v-\mathrm{X}_v\mathrm{Y}_u),\\
\mathrm{E} &= g\cos\theta_0(\mathrm{X}_v\mathrm{G}_u-\mathrm{X}_u\mathrm{G}_v)-g\sin\theta_0(\mathrm{Y}_v\mathrm{G}_u-\mathrm{Y}_u\mathrm{G}_v),
\end{aligned}\right\} \quad (5).$$

In many cases it is possible to take the axes of co-ordinates, so that θ_0 shall be zero, and also that Y and its differentials shall vanish. In such cases the coefficients take the following simple forms:—

$$\left.\begin{aligned}
\mathrm{A} &= k^2,\\
\mathrm{B} &= k^2\mathrm{X}_u+\mathrm{G}_{\dot\theta},\\
\mathrm{C} &= v_0\mathrm{G}_u-u_0\mathrm{G}_v-\mathrm{X}_{\dot\theta}\mathrm{G}_u,\\
\mathrm{D} &= u_0(\mathrm{X}_v\mathrm{G}_u-\mathrm{X}_u\mathrm{G}_v)-g\mathrm{G}_v,\\
\mathrm{E} &= g(\mathrm{X}_v\mathrm{G}_u-\mathrm{X}_u\mathrm{G}_v),
\end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\quad (6).$$

Equation (4) or (4*a*) is the period equation for small fluctuations about steady motion.

In order that the steady motion may be stable, the roots of this biquadratic must either be real and negative, or complex with their real parts negative, and Routh* finds that this will be the case if the six quantities

$$A,\ B,\ C,\ D,\ E,\ \text{and}\ BCD - AD^2 - EB^2,$$

are all of the same sign. Since A is essentially positive the remaining five quantities must all be positive. We shall denote the last quantity by H.

3. *General Theorems.*†

(1) The general transformation formulæ, connecting the nine coefficients $X_u \ldots G_{\dot\theta}$ referred to any given system of axes with those referred to any other system of axes, may be easily written down and need not be discussed here.

(2) The work of calculating these coefficients for a given system may be reduced if the resistance is proportional to the square of the velocity, for X, Y, G will then be homogeneous quadratic functions of u, v, $\dot\theta$, and Euler's theorem of homogeneous functions gives, remembering that $\dot\theta_0 = 0$ and applying (2),

$$uX_u + vX_v = 2g\cos\theta,$$

$$uY_u + vY_v = -2g\sin\theta,$$

$$uG_u + vG_v = 0.$$

(3) If V is the velocity of gliding, the coefficients $mX_u \ldots mG_{\dot\theta}$ are all linear functions of V for a given angle of gliding. But mg being, in steady motion, equal to the vertical resistance, is proportional to V^2. Hence if m be eliminated, the values of $X_u \ldots G_{\dot\theta}$ are inversely proportional to V. In this case

A is independent of V,
B is of dimensions V^{-1},
C is of form $P + QV^{-2}$,
D is of dimensions V^{-1},
E is of dimensions V^{-2},

and the expression H or $BCD - AD^2 - EB^2$ assumes the form $PV^{-2} + QV^{-4}$. Thus the conditions of stability $C > O$ and $H > O$ impose limits on the value of V^2, the remaining conditions only depend on the form and dimensions of the machine and the angle of gliding.

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* Routh, 'Advanced Rigid Dynamics,' p. 167.

† This section and also the subsequent parts enclosed in [......] have been rewritten October 27, 1903.

We shall now show how the nine coefficients $X_u \dots G_{\dot\theta}$ can be calculated for a system of aeroplanes if the laws of variation of the resultant pressures and of the positions of the centres of pressure are known. We shall assume that the resultant pressure on a plane lamina varies as the square of the velocity, and acts in a direction normal to the lamina.

We may therefore write

$$R = KSV^2 f(\alpha),$$

where S is the area of the lamina, V the velocity, and α the angle between the plane of the lamina and the direction of motion, K a constant depending on the units employed.

The function $f(\alpha)$ has been determined by Langley* for certain rectangular planes.

Also, let the distance of the centre of pressure from the centre of figure be $a\phi(\alpha)$, $2a$ being the breadth of the lamina.

Experiments to determine $\phi(\alpha)$ for square planes have been made by Joesel,† Kummer,‡ and Langley,§ and Kummer has also experimented on oblong planes, with their longer side in the direction of motion.

Their results show a certain amount of discrepancy, and there appears to be considerable difficulty in obtaining consistent results at small inclinations.

Further experiments are very necessary, and it is to be hoped that more attention will be given to determinations of $\phi(\alpha)$, when their importance, as affecting the stability, has been recognised.

Experiments show that $f(\alpha)$, $\phi(\alpha)$ are, to a first approximation, independent of the translational velocity of the lamina, but the effects of a rotational angular velocity, $\dot\theta$, have never been considered. It would not be difficult to determine these effects by experiments with a whirling table, making $\dot\theta$ the angular velocity of the table. Failing such experiments, these effects must be neglected, and, as $\dot\theta$ is zero in the steady motion, and only small oscillations are considered, they are probably small. When the system consists of a number of planes rigidly connected together, we may now write the component forces and couple in the form

$$\left.\begin{aligned} mX &= \Sigma KS_1V_1^2 f(\alpha_1-\beta_1)\cos\beta_1 \\ mY &= \Sigma KS_1V_1^2 f(\alpha_1-\beta_1)\sin\beta_1 \\ mG &= \Sigma KS_1V_1^2 f(\alpha_1-\beta_1)\{p_1+a_1\phi(\alpha_1-\beta_1)\} \end{aligned}\right\} \dots\dots \quad (7),$$

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* 'Experiments in Aerodynamics,' p. 62.

† Joesel, 'Mémorial du Genie Maritime,' 1870.

‡ 'Berlin Akad. Abhandlungen,' 1875, 1876.

§ *Loc. cit.*, p. 90.

where S_1 is the area, $2a_1$ the breadth of the lamina, ξ_1, η_1 the co-ordinates of its centre, V_1 the resultant velocity of its centre, $\alpha_1 = \tan^{-1} u_1/v_1$ the angle which the direction of motion makes with the axis of y, β_1 the angle which the plane of the lamina makes with the axis of y, so that $(\alpha_1 - \beta_1)$ is the angle between the plane and the direction of motion of the centre of mass, and finally, $p_1 = \eta_1 \cos \beta_1 + \xi_1 \sin \beta_1$.

The summation is to be extended over all the planes of the system.

If, now, u, v be the component velocities of the centre of gravity the velocities u_1, v_1 of the centre of the lamina will be given by

$$u_1 = u - \eta_1 \dot{\theta}, \qquad v_1 = v + \xi_1 \dot{\theta},$$

and for steady motion, since $\dot{\theta} = 0$, $u_1 = u$, $v_1 = v$.

We may therefore write the expressions for the forces in the form

$$\begin{aligned} mX &= \Sigma KS_1 (u_1^2 + v_1^2) f(\tan^{-1} u_1/v_1 - \beta_1) \cos \beta_1 \\ &= \Sigma KS_1 (u^2 + v^2 - 2u\eta_1\dot{\theta} + 2v\xi_1\dot{\theta}) f \left\{ \tan^{-1} \left(\frac{u - \eta_1 \dot{\theta}}{v + \xi_1 \dot{\theta}} \right) - \beta_1 \right\} \cos \beta_1, \end{aligned}$$

neglecting terms in $\dot{\theta}^2$.

Similarly,

$$mY = \Sigma KS_1 (u^2 + v^2 - 2u\eta_1\dot{\theta} + 2v\xi_1\dot{\theta}) f \left\{ \tan^{-1} \left(\frac{u - \eta_1 \dot{\theta}}{v + \xi_1 \dot{\theta}} \right) - \beta_1 \right\} \sin \beta_1,$$

$$\begin{aligned} mG = &- \Sigma KS_1 (u^2 + v^2 - 2u\eta_1\dot{\theta} + 2u\xi_1\dot{\theta}) f \left\{ \tan^{-1} \left(\frac{u - \eta_1 \dot{\theta}}{v + \xi_1 \dot{\theta}} \right) - \beta_1 \right\} \\ &\left\{ p_1 + a\phi \left(\tan^{-1} \left(\frac{u - \eta_1 \dot{\theta}}{v + \xi_1 \dot{\theta}} \right) - \beta_1 \right) \right\}. \end{aligned}$$

Differentiating these expressions with respect to u, v, $\dot{\theta}$ we have :—

$$mX_u = \Sigma \left[KS_1 (2u - \eta_1 \dot{\theta}) f(\alpha_1 - \beta_1) \cos \beta_1 + KS_1 v_1 f'(\alpha_1 - \beta_1) \cos \beta_1 \right];$$

or, since $\dot{\theta} = 0$ in the steady motion,

$$mX_u = \Sigma \left[2KS_1 u f(\alpha_1 - \beta_1) \cos \beta_1 + KS_1 v f'(\alpha_1 - \beta_1) \cos \beta_1 \right] \ldots. \quad (8),$$

Similarly,

$$mX_v = \Sigma \left[2KS_1 v f(\alpha_1 - \beta_1) \cos \beta_1 - KS_1 u f'(\alpha_1 - \beta_1) \cos \beta_1 \right],$$

$$mY_u = \Sigma \left[2KS_1 u f(\alpha_1 - \beta_1) \sin \beta_1 + KS_1 v f'(\alpha_1 - \beta_1) \sin \beta_1 \right],$$

$$mY_v = \Sigma \left[2KS_1 v f(\alpha_1 - \beta_1) \sin \beta_1 - KS_1 u f'(\alpha_1 - \beta_1) \sin \beta_1 \right],$$

$$\begin{aligned} mX_{\dot{\theta}} = \Sigma \big[& 2KS_1 (-u\eta_1 + v\xi_1) f(\alpha_1 - \beta_1) \cos \beta_1 \\ & - KS_1 (v\eta_1 + u\xi_1) f'(\alpha_1 - \beta_1) \cos \beta_1 \big], \end{aligned}$$

$$mY_{\dot\theta} = \Sigma\,[2KS_1\,(-u\eta_1+v\xi_1)\,f(\alpha_1-\beta_1)\sin\beta_1 \\ \qquad -KS_1\,(v\eta_1+u\xi_1)\,f'(\alpha_1-\beta_1)\sin\beta_1],$$

$$mG_u = \Sigma\,[-2KS_1\,(u-\eta_1\dot\theta)\,f(\alpha_1-\beta_1)\{p_1+a_1\phi\,(\alpha_1-\beta_1)\} \\ -KS_1v_1\,f'(\alpha_1-\beta_1)\{p_1+a_1\phi\,(\alpha_1-\beta_1)\} \\ -KSv_1\,f(\alpha_1-\beta_1)\,a_1\phi'(\alpha_1-\beta_1)].$$

Since in the steady motion $G = 0$,

$$i.e.,\quad 2\,(u^2+v^2)\,\Sigma KS_1\,f(\alpha_1-\beta_1)\,\{p_1+a_1\phi\,(\alpha_1-\beta_1)\} = 0,$$

$$mG_u = v\Sigma\,[-KS_1\,f'(\alpha_1-\beta_1)\,\{p_1+a_1\phi\,(\alpha_1-\beta_1)\} \\ \qquad -KS_1\,f(\alpha_1-\beta_1)\,a_1\phi'(\alpha_1-\beta_1],$$

$$mG_v = u\Sigma\,[KS_1\,f'(\alpha_1-\beta_1)\,\{p_1+a_1\phi\,(\alpha_1-\beta_1)\} \\ \qquad +KS_1\,f(\alpha_1-\beta_1)\,a_1\phi'(\alpha_1-\beta_1)],$$

$$mG_{\dot\theta} = \Sigma\,[-KS_1\,f'(\alpha_1-\beta_1)\,(-v\eta_1+u\xi_1)\,\{p+a\phi\,(\alpha_1-\beta_1)\} \\ \qquad -KS_1\,f(\alpha_1-\beta_1)\,(-v\eta_1+u\xi_1)\,a\phi'(\alpha_1-\beta_1)].$$

4. *Numerical Calculations for Particular Cases. Single Laminæ.*

We now consider the stability of certain particular systems, beginning with a single plane lamina.

Since the lamina is falling steadily under gravity, its plane must necessarily be horizontal.

Take the axis of y parallel to this plane, then $\theta = 0$, $\beta = 0$, and, therefore, $Y = 0$; also, for equilibrium, $\eta + a\phi\,(\alpha) = 0$.

The coefficients, therefore, become

$$mX_u = 2KSuf(\alpha)+KSvf'(\alpha),$$

$$mX_v = 2KSvf(\alpha)-KSuf'(\alpha),$$

$$mX_{\dot\theta} = -2KS\,(u\eta-v\xi)\,f(\alpha)+KSf'(\alpha)\,(v\eta+u\xi),$$

$$mG_u = -KSvf(\alpha)\,a\,\phi'(\alpha),$$

$$mG_v = KSuf(\alpha)\,a\phi'(\alpha),$$

$$mG_{\dot\theta} = -KSaf(\alpha)\,(v\eta+u\xi)\,a\phi'(\alpha).$$

Ex. 1.—Consider a square plane, balanced so as to fly at an angle of 10°, with the centre of gravity in the plane, and suppose its radius of gyration given by $k^2 = \frac{1}{2}a^2$.

Professor Langley's results give for this particular angle

$$f(\alpha) = 0{\cdot}3, \qquad f'(\alpha) = 1{\cdot}6,$$

and from Joesel's formula

$$\phi(\alpha) = 0{\cdot}49, \qquad \phi'(\alpha) = -0{\cdot}59, \qquad \therefore\ \eta = 0{\cdot}49a.$$

also

$$u = \mathrm{V} \sin \alpha = 0{\cdot}17\mathrm{V}, \qquad v = 0{\cdot}98\mathrm{V}.$$

Substituting these values in the above expressions for X_u . . ., we have

$$m\mathrm{X}_u = 1{\cdot}66\mathrm{KSV}, \qquad m\mathrm{G}_u = 0{\cdot}17\mathrm{KSV}a,$$
$$m\mathrm{X}_v = 0{\cdot}32\mathrm{KSV}, \qquad m\mathrm{G}_v = -0{\cdot}03\mathrm{KSV}a,$$
$$m\mathrm{X}_{\dot\theta} = 0{\cdot}73\mathrm{KSV}a, \qquad m\mathrm{G}_{\dot\theta} = 0{\cdot}09\mathrm{KSV}a^2.$$

Substituting these values in the expressions for A, B, C, D, E, given in § (2), and remembering that by the conditions for steady motion $mg = \mathrm{KSV}^2 f(\alpha)$, we obtain

$$\mathrm{A} = \tfrac{1}{2}a^2, \qquad \mathrm{B} = 92\, a^2/\mathrm{V}, \qquad \mathrm{C} = 18a - 1340\, a^2/\mathrm{V}^2,$$
$$\mathrm{D} = 295\, a/\mathrm{V}, \qquad \mathrm{E} = 36400\, a/\mathrm{V}^2,$$

and therefore

$$\mathrm{H} \equiv \mathrm{BCD} - \mathrm{AD}^2 - \mathrm{EB}^2 = 4{\cdot}8 \,.\, 10^5 \frac{a^4}{\mathrm{V}^2} - 3{\cdot}6 \,.\, 10^7 \frac{a^5}{\mathrm{V}^4}$$
$$- 4{\cdot}3 \,.\, 10^4 \frac{a^4}{\mathrm{V}^2} - 3{\cdot}08 \,.\, 10^8 \frac{a^5}{\mathrm{V}^4}.$$

This expression will be positive if $\mathrm{V}^2 > 774a$, and this condition will also make C positive. The glider will therefore be stable if its velocity is greater than $\sqrt{(774a)}$, the units being feet and seconds.

Ex. 2.—Let us now take the angle of gliding to be 35°, and assume as before $k^2 = \frac{1}{2} a^2$. At this angle we have

$$f(\alpha) = 0{\cdot}84, \qquad f'(\alpha) = 0{\cdot}6,$$
$$\phi(\alpha) = 0{\cdot}26. \qquad \phi'(\alpha) = 0{\cdot}5.$$

With these values, we have

$$m\mathrm{X}_u = 1{\cdot}34\mathrm{KSV}, \qquad m\mathrm{G}_u = 0{\cdot}35\mathrm{KSV}a,$$
$$m\mathrm{X}_v = 1{\cdot}1\mathrm{KSV}, \qquad m\mathrm{G}_v = 0{\cdot}21\mathrm{KSV}a,$$
$$m\mathrm{X}_{\dot\theta} = -0{\cdot}1\mathrm{KSV}a, \qquad m\mathrm{G}_{\dot\theta} = 0{\cdot}08\mathrm{KSV}a.$$

Substituting in the expressions for A, B, etc., we have

$$\mathrm{A} = \tfrac{1}{2}a^2, \qquad \mathrm{B} = 27\, a^2/\mathrm{V},$$
$$\mathrm{C} = 15{\cdot}2a + 50\, a^2/\mathrm{V}^2, \qquad \mathrm{D} = 380\, a/\mathrm{V} + 240\, a/\mathrm{V} = 620\, a/\mathrm{V},$$
$$\mathrm{E} = 25600\, a^2/\mathrm{V}^2, \qquad \mathrm{H} = 5600\, a^3/\mathrm{V}^2 - 17400000\, a^5/\mathrm{V}^4.$$

The latter is positive when $\mathrm{V}^2 > 3100a$, which is therefore the condition of stability.

Ex. 3.—Let us now take an oblong plane lamina; the values of $f(\alpha)$

are given by Langley for a lamina of 30″ by 4·8″ the shorter side being in the plane of motion. In the absence of any information as to $\phi(\alpha)$, we shall assume (whether correctly or incorrectly) the same values as for square planes.

Let the moment of inertia be given, as before, by $k^2 = \frac{1}{2}a^2$.

For an angle of 10° we have

$$f(\alpha) = 0{\cdot}44, \qquad f'(\alpha) = 1{\cdot}5,$$
$$\phi(\alpha) = 0{\cdot}49, \qquad \phi'(\alpha) = 0{\cdot}59.$$

Substituting these values in the expressions for the coefficients, we have

$$mX_u = 1{\cdot}6\text{KSV}, \qquad mG_u = 0{\cdot}26\text{KSV}a,$$
$$mX_v = 0{\cdot}6\text{KSV}, \qquad mG_v = -0{\cdot}4\text{KSV}a,$$
$$mX_{\dot\theta} = 0{\cdot}56\text{KSV}a, \qquad mG_{\dot\theta} = 0{\cdot}13\text{KSV}a^2.$$

Substituting again in the expressions for A, B, C, D, and E, we have, assuming $k^2 = \frac{1}{2}a^2$,

$$\text{A} = k^2 = \tfrac{1}{2}a^2,$$
$$\text{B} = k^2X_u + G_{\dot\theta} = \frac{1}{m}\left(1{\cdot}6\frac{a^2}{2} + 0{\cdot}13a^2\right)\text{KSV} = 60a^2/\text{V},$$
$$\text{C} = vG_u - uG_v - X_{\dot\theta}\,G_u = 19{\cdot}4a - 760a^2/\text{V}^2,$$
$$\text{D} = u(X_vG_u - X_uG_v) - gG_v = 280a/\text{V},$$
$$\text{E} = g(X_vG_u - X_uG_v) = 35000a/\text{V}^2.$$

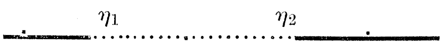
Therefore

$$\text{H} = 3{\cdot}2 \,.\, 10^5\frac{a^4}{\text{V}^2} - 1{\cdot}2 \,.\, 10^7\frac{a^5}{\text{V}^4} - 3{\cdot}9 \,.\, 10^4\frac{a^4}{\text{V}^2} - 1{\cdot}2 \,.\, 10^8\frac{a^5}{\text{V}^4}\,.$$

This is positive if $\text{V}^2 > 470a$, which is therefore the condition of stability.

5. *Gliders Formed of two Planes.*

Proceeding now to consider the stability of systems made up of several planes rigidly connected together, we shall first consider the stability of a gliding system supported on two slats S_1 S_2, which are so narrow that displacements of their centre of pressures due to variations of the angles of incidence of the wind may be neglected.



We shall first suppose that the two slats are in the same plane, and that the centre of gravity is also in this plane.

Putting $\phi(\alpha) = 0$ and $\beta_1 = \beta_2 = 0$ in the expressions of § (3), we have all the Y's equal to zero and

$$\begin{aligned} mX_u &= K(S_1 + S_2)(2uf(\alpha) + vf'(\alpha)), \quad \ldots\ldots\ldots\ldots\ldots \quad (9) \\ mX_v &= K(S_1 + S_2)(2vf(\alpha) + uf'(\alpha)), \\ mX_{\dot\theta} &= -K(S_1\eta_1 + S_2\eta_2)(2uf\alpha + vf'\alpha), \\ mG_u &= -Kvf'(\alpha)(S_1\eta_1 + S_2\eta_2), \\ mG_v &= Kuf'(\alpha)(S_1\eta_1 + S_2\eta_2), \\ mG_{\dot\theta} &= Kvf'(\alpha)(S_1\eta_1^2 + S_2\eta_2^2). \end{aligned}$$

Now for equilibrium we must have

$$S_1\eta_1 + S_2\eta_2 = 0.$$

In this case, G_u, G_v, and $X_{\dot\theta}$ vanish, and therefore, from (6), we see that the coefficients C, D, E, will be zero, and the equilibrum is critical or neutral.

[This result, which is also evident from first principles, holds good equally when the centre of gravity is not in the plane of the laminæ. In order to make the system really stable, the laminæ must be inclined at small angles to the line joining their centres.

Calling these angles β_1 and β_2, neglecting $\phi(\alpha)$, and writing $\alpha' = \alpha - \beta_1$, $\alpha'' = \alpha - \beta_2$, we get in the equations (8),

$$\begin{aligned} mX_u &= K\Sigma S_1(2uf(\alpha') + vf'(\alpha'))\cos\beta_1, \\ mX_v &= K\Sigma S_1(2vf(\alpha') - uf'(\alpha'))\cos\beta_1, \\ mX_{\dot\theta} &= -K\Sigma S_1\eta_1(2uf(\alpha') + vf'(\alpha'))\cos\beta_1, \\ mY_u &= K\Sigma S_1(2uf(\alpha') + vf'(\alpha'))\sin\beta_1, \\ mY_v &= K\Sigma S_1(2vf(\alpha') - uf'(\alpha'))\sin\beta_1, \\ mY_{\dot\theta} &= -K\Sigma S_1\eta_1(2uf(\alpha') + vf'(\alpha'))\sin\beta_1, \\ mG_u &= -Kv\Sigma S_1 p_1 f'(\alpha'), \\ mG_v &= Ku\Sigma S_1 p_1 f'(\alpha'), \\ mG_{\dot\theta} &= Kv\Sigma S_1\eta_1 p_1 f'(\alpha'). \end{aligned}$$

The conditions for steady motion give

$$\begin{aligned} 0 &= kS_1 f(\alpha')\sin\beta_1 + kS_2 f(\alpha'')\sin\beta_2, \\ mg/V^2 &= kS^1{}_1 f(\alpha')\cos\beta_1 + kS_2 f(\alpha'')\cos\beta_2, \\ 0 &= kS_1 f(\alpha')\,\eta_1\cos\beta + kS_2 f(\alpha'')\,\eta_2\cos\beta_2. \end{aligned}$$

With these substitutions, those of the nine coefficients $X_u \ldots G_{\dot\theta}$ which vanish in the limiting case of coplanar laminæ are given by

$$\frac{X_{\dot\theta}}{v} = \frac{G_u}{v} = -\frac{G_v}{u} = -\frac{g}{V^2}\frac{\eta_2\eta_1}{\eta_2-\eta_1}\left\{\frac{f'(\alpha')}{f(\alpha')} - \frac{f'(\alpha'')}{f(\alpha'')}\right\},$$

$$\frac{Y_u}{v} = -\frac{Y_v}{u} = \frac{g}{V^2}\frac{\sin\beta_1\sin\beta_2}{\sin(\beta_2-\beta_1)}\left\{\frac{f'(\alpha')}{f(\alpha')} - \frac{f'(\alpha'')}{f(\alpha'')}\right\},$$

and an expression for $Y_{\dot\theta}$, which is not required.

The remaining coefficients may for a first approximation be taken the same as in (9). We notice the following points.

(1) The expressions for $X_{\dot\theta}$, Y_u, Y_v, G_u and G_v contain the factor $f'(\alpha')/f(\alpha'') - f'(\alpha'')/f(\alpha'')$. In the laws of resistance commonly assumed $f'(\alpha)/f(\alpha)$ decreases as α increases, and this coefficient, therefore, becomes negative if $\alpha' < \alpha''$, *i.e.*, if the two planes slope towards each other α' referring to the front plane. This is the case in a gliding machine furnished with a rudder, if the rudder is tilted slightly upwards. In such cases η_2 must be negative, hence $X_{\dot\theta}$ and G_u are positive, Y_u and G_u are negative.

(2) If β_1 and β_2 are of the first order of small quantities, $\beta_1 - \beta_2$ also being of the first order, the expressions $X_{\dot\theta}$, G_u and G_v are of the first order, but Y_u and Y_v are of the second order, and hence equations (6) give approximately the five coefficients A, B, C, D, E.

(3) The fifth stability condition $H > 0$ appears at first sight difficult to reconcile with the smallness of the coefficients C, D, E for small inclinations of the planes, as B^2E is of a lower order of small quantities than BCD. To satisfy this fifth condition, however, the important thing is to make E small compared with C. Now C will be found to consist of two parts, one independent of V and the other negative and proportional to $1/V^2$, while E is proportional to $1/V^2$, hence stability can best be secured by making V^2 sufficiently large.]

(4) If the planes are not infinitely narrow, it will be found that X_u, X_v, Y_u, Y_v are the same as before, and $-mG_u/v$, and mG_v/u are both increased by $\Sigma KS_1a_1(f'(\alpha')\phi(\alpha') + f(\alpha')\phi'(\alpha'))$. This will have the effect of altering C, E, D in the same proportion, and will, therefore, not alter the stability, except when $\alpha = 0$, when the additional term will prevent E_1D from becoming zero. $G_{\dot\theta}$ is increased by the term $v\Sigma KS_1a_1\eta_1(f'(\alpha')\phi(\alpha') + f(\alpha')\phi'(\alpha))$. This term will, in general, be negative, and therefore $G_{\dot\theta}$ will be diminished, which will have the effect of diminishing B.

6. *Examples of Two-Plane Gliders.*

We shall now assume certain particular values for the dimensions of gliders of this form, and proceed to calculate the conditions of stability.

Ex. 4.—Let us first consider the case of the two slats of equal area set at angles such that

$$\alpha_1 = 15^\circ, \qquad \alpha_2 = 5^\circ.$$

Then we have from Langley's results,

$$f(\alpha') = 0{\cdot}56, \qquad f'(\alpha') = 1,$$

$$f(\alpha'') = 0{\cdot}28, \qquad f'(\alpha'') = 2{\cdot}5\,;$$

for equilibrium

$$\alpha' f(\alpha'') = -\eta_2 f(\eta_1)\,;$$

therefore, if we put $\eta_1 = a$, we have $\eta_2 = 2a$. Let k^2 be assumed $= 2\alpha^2$. Substituting in the expressions for X_u, etc., we have

$$mX_u = 3{\cdot}6\mathrm{KSV}, \qquad mG_u = 4\mathrm{KSV}a,$$

$$mX_v = 1\mathrm{KSV}, \qquad mG_v = 0{\cdot}7\mathrm{KSV}a,$$

$$mX_{\dot\theta} = 4\mathrm{KSV}a, \qquad mG_{\dot\theta} = 11\mathrm{KSV}a^2,$$

and therefore,

$$A = 2a^2, \qquad B = 680\frac{a^2}{V}, \qquad C = 156a - 22800a^2/V^2,$$

$$D = 2721\frac{a}{V}, \qquad E = 297000\frac{a}{V^2}, \qquad H = 2.7\,.\,10^8\frac{a^4}{V^2} - 1{\cdot}4\,.\,10^7\frac{a^4}{V^2}$$

$$- 2{\cdot}9\,.\,10^{10}\frac{a^3}{V^4} - 1{\cdot}3\,.\,10^{11}\frac{a^3}{V^4}.$$

This is positive if $V^2 > 590a$.

Ex. 5.—Let us now take the case of two equal square planes inclined at a small angle to each other.

Let $\alpha = 10^\circ, \quad \beta_1 = -5^\circ, \quad \beta_2 = 5^\circ,$

and let $2a$ be the breadth of either plane.

For these angles we have

$$f(\alpha_1 - \beta_1) = 0{\cdot}44, \qquad f'(\alpha_1 - \beta_1) = 1{\cdot}7,$$

$$f(\alpha_2 - \beta_2) = 0{\cdot}15, \qquad f'(\alpha_2 - \beta_2) = 1{\cdot}8,$$

$$\phi(\alpha_1 - \beta_1) = 0{\cdot}45, \qquad \phi'(\alpha_1 - \beta_1) = 0{\cdot}6,$$

$$\phi(\alpha_2 - \beta_2) = 0{\cdot}55, \qquad \phi'(\alpha_1 - \beta_2) = 0{\cdot}58.$$

For equilibrium we must have

$$f(\alpha_1 - \beta_1)\{\eta_1 + a\phi(\alpha_1 - \beta_1)\} = f(\alpha_2 - \beta_2)\{\eta_2 + a\phi(\alpha_2 - \beta_2)\},$$

whence

$$3\eta_1 + \eta_2 = 0{\cdot}28a\,;$$

therefore, if we put $\eta_2 = -4a$, we have $\eta_1 = 1{\cdot}4a$. Let k^2 be assumed to be $= 4a^2$. Substituting these values in the expressions for the coefficients, we have

$$mX_u = 3{\cdot}6\text{KSV}, \qquad mY_u = 0{\cdot}02\text{KSV},$$

$$mX_v = 0{\cdot}58\text{KSV}, \qquad mY_v = -0{\cdot}1\text{KSV},$$

$$mX_{\dot\theta} = 1{\cdot}8\text{KSV}a, \qquad mY_{\dot\theta} = 0{\cdot}00\text{KSV},$$

$$mG_u = 2{\cdot}61\text{KSV}a, \qquad mG_v = 0{\cdot}44\text{KSV}a,$$

$$mG_{\dot\theta} = 30\text{KSV}a^2.$$

Substituting again in the expressions for A, B, C, D, E, we obtain, putting $k^2 = 4a^2$,

$$A = 4a^2, \qquad B = 2390\frac{a^2}{V}, \qquad C = 142a - 13700\frac{a^2}{V^2},$$

$$D = 2500\frac{a}{V}, \qquad E = 284000\frac{a}{V^2}.$$

H is positive if $V^2 > 2000a$, which is the condition of stability for this form of glider.

By putting $l = 7{\cdot}4a$, l represents approximately the extreme length of the glider, and the condition of stability reduces to $V^2 > 270l$.

Ex. 6.—Consider next the case of two unequal square planes, inclined at an angle, and in the first case suppose that the smaller plane is in front.

Let S and S′ be the respective areas, and suppose $S = 10S'$.

Let $2a$ be the breadth of the large plane, and $2a/3{\cdot}1$ that of the smaller, also let the distance between the centres be equal to $3a$; and let $k^2 = a^2$.

Let the angle α between the large plane and the direction of motion be 10°, and let the angle between the two planes be also 10°, so that the small plane is inclined at an angle of 20° to the direction of motion.

Then we have

$$f(\alpha) = 0{\cdot}3, \qquad f'(\alpha) = 1{\cdot}6, \qquad \phi(\alpha) = 0{\cdot}49, \qquad \phi'(\alpha) = 0{\cdot}59,$$

$$f(\alpha+\beta) = 0{\cdot}5, \quad f'(\alpha+\beta) = 1{\cdot}57, \quad \phi(\alpha+\beta) = 0{\cdot}4, \quad \phi'(\alpha+\beta) = 0{\cdot}56.$$

For equilibrium we have, if η be the distance between c.g. and centre of large plane,

$$Sf(\alpha)\,(\eta - a\phi(\alpha)) = S'f(\alpha+\beta)\,(b + a'\,\phi(\alpha+\beta) - \eta).$$

Therefore $\eta = 0{\cdot}94a$, so that $\eta_1 = b - \eta = 2{\cdot}06a$.

Substituting in the expressions for X_u, etc.,

$$mX_u = 1{\cdot}87\ \text{KSV}, \qquad mX_v = 0{\cdot}41\ \text{KSV}, \qquad mX_{\dot\theta} = 1{\cdot}4\ \text{KSV}a,$$

$$mG_u = 0{\cdot}57\ \text{KSV}a, \qquad mG_v = 0{\cdot}097\ \text{KSV}a, \qquad mG_{\dot\theta} = 1{\cdot}13\ \text{KSV}a^2.$$

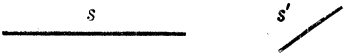
Substituting these values in the expressions for A, B, etc., assuming that $k^2 = a^2$, we have

$$A = a^2, \qquad B = 273\frac{a^2}{V}, \qquad C = 50{\cdot}8a - 15050\frac{a^2}{V^2},$$

$$D = 917\frac{a}{V}, \qquad E = 120000\frac{a}{V^2}, \qquad H = 118\,.\,10^5\frac{a^4}{V^2} - 123\,.\,10^8\frac{a^5}{V^4}.$$

This will be positive if $V^2 > 1040a$. This condition will also make C positive, and is therefore the condition of stability.

Ex. 7.—Let us now suppose the small plane to be placed behind the other



being inclined to it at an angle of 5°, the direction of motion again making an angle of 10° with the large plane and k^2 being $= a^2$, as in Ex. 6.

In this case we have

$$f(\alpha) = 0{\cdot}3, \quad f'(\alpha) = 1{\cdot}6, \quad \phi(\alpha) = 0{\cdot}49, \quad \phi'(\alpha) = 0{\cdot}59,$$
$$f(\alpha') = 0{\cdot}17, \quad f'(\alpha') = 1{\cdot}8, \quad \phi(\alpha') = 0{\cdot}55, \quad \phi'(\alpha') = 0{\cdot}6.$$

If the distance between the centres of the planes be $3a$, the conditions of equilibrium give

$$\eta = -0{\cdot}35a, \qquad \eta_1 = -2{\cdot}65a.$$

Substituting, we obtain

$$mX_u = 1{\cdot}9\,\text{KSV}a, \qquad mG_u = 0{\cdot}33\,\text{KSV}a,$$
$$mX_v = 0{\cdot}41\,\text{KSV}a, \qquad mG_v = 0{\cdot}056\,\text{KSV}a,$$
$$mX = 0{\cdot}65\,\text{KSV}a, \qquad mG_{\dot\theta} = 1{\cdot}32\,\text{KSV}a^2.$$

$$\therefore\ A = a^2, \qquad B = 233\frac{a^2}{V}, \qquad C = 34{\cdot}8a - 2400\frac{a^2}{V^2},$$

$$D = 1780\frac{a}{V}, \qquad E = 80000\frac{a}{V^2}, \qquad H = 12\,.\,10^6\frac{a^4}{V^2} - 3\,.\,10^9\frac{a^5}{V^4}.$$

H will be positive if $V^2 > 250a$.

The condition of stability is therefore that $V > \surd(250a)$.

[7. *Effect of Moment of Inertia on Stability.*

It will be seen from (5) that the radius of gyration, k^2, occurs only in the expressions for A, B, the first two coefficients of the determinantal equation. We have $A = k^2$, $B = k^2(X_u + Y_v) + G_{\dot\theta}$. In all the cases considered $X_u + Y_v$ and $G_{\dot\theta}$ are positive, and therefore these two coefficients are positive for all values of k^2.

Taking the expression

$$\mathrm{H} = \mathrm{BCD} - \mathrm{AD}^2 - \mathrm{EB}^2,$$

it may be written

$$\mathrm{CDG}_{\dot\theta} - \mathrm{EG}_{\dot\theta}^2 + k^2\{(\mathrm{X}_u + \mathrm{Y}_v)\,\mathrm{CD} - \mathrm{D}^2 - 2\,(\mathrm{X}_u + \mathrm{Y}_v)\mathrm{G}_{\dot\theta}\mathrm{E}\} - k^4(\mathrm{X}_u + \mathrm{Y}_v)^2\mathrm{E}.$$

H is therefore increased with increase of k^2 if

$$(\mathrm{X}_u + \mathrm{Y}_v)\,(\mathrm{CD} - 2\mathrm{G}_{\dot\theta}\mathrm{E}) - \mathrm{D}^2 - 2k^2\,(\mathrm{X}_u + \mathrm{Y}_v)\,\mathrm{E} > 0.$$

At the critical velocity, if $\mathrm{H} = 0$, this becomes

$$\mathrm{CDG}_{\dot\theta} + (\mathrm{X}_u + \mathrm{Y}_v)^2\mathrm{E}/k^4 < \mathrm{EG}_{\dot\theta}^2.$$

Unless this condition is fulfilled, the critical velocity given by $\mathrm{H} = 0$ increases with increase of k^2.

In all the numerical examples considered above, $\mathrm{CDG}_{\dot\theta} > \mathrm{EG}_{\dot\theta}^2$ and E is positive, so that the critical velocity is increased by increasing k^2. Thus in Ex. (1) k^2 was taken equal to $\frac{1}{2}a^2$, and the critical velocity obtained was $\surd(774a)$; if, instead, we had taken $k^2 = a^2$, the critical velocity would have been $\surd(1240a)$.—*Jan.*, 1904.]

8. *Character of the Fluctuations about Steady Motion. Mode in which the System Overturns.*

The character of the fluctuations about steady motion depends on the nature of the roots of the biquadratic (4*a*), the expressions for the displacements δu, δv, $\delta\theta$ being evidently of the following forms:—

For roots all real	$c_1e^{\lambda_1 t} + c_2e^{\lambda_2 t} + c_3e^{\lambda_3 t} + c_4e^{\lambda_4 t}$.
For roots two real, two imaginary ...	$c_1e^{\lambda_1 t} + c_2e^{\lambda_2 t} + \gamma e^{\alpha t}\cos(\beta t - \epsilon)$.
For two pairs of imaginary roots	$\gamma_1 e^{\alpha_1 t}\cos(\beta_1 t - \epsilon_1)$ $+ \gamma_2 e^{\alpha_2 t}\cos(\beta_2 t - \epsilon_2)$.

The last form indicates two different sets of undulations of different lengths. Photographs of the paths of gliders taken by magnesium light distinctly show these two undulations, thus confirming our theory.

An important further consequence is that a glider may perform undulations decreasing in amplitude, corresponding to a pair of complex roots of the biquadratic with their real part negative, but the motion may be unstable through the other roots having their real part positive, or one or both of them being real and positive. This indicates a real danger in experimenting with gliders.

Stability may be broken either if a real root of the equation (4*a*) changes from negative to positive, or if the real part of a pair of imaginary roots changes from negative to positive.

The condition for the latter is that $\mathrm{H} = 0$, whereas a real root changes sign if $E = 0$. If stability is broken by a fall of velocity the only quantities which can vanish are H and C, of these H is more difficult to make positive than C, and hence it appears that the most likely way for a glider to overturn in general is by commencing with a series of oscillations of increasing amplitude. This again agrees with experiments.

9. *Conclusions.*

1. For a glider or other body moving in a vertical plane in a resisting medium of any kind whatever, the small oscillations about a state of uniform rectilinear motion are determined by an equation of the fourth degree, so that the conditions for stable steady motion are those obtained by Routh.

2. The coefficients in the period equation involve, in addition to the ordinary dynamical constants, nine quantities $X_u \ldots G_{\dot{\theta}}$, which, when referred to rectangular axes fixed in the body, represent the differential coefficients of the forces and couple due to the aerial resistances with respect to its translatory and rotatory velocity components.

3. In the case of a system of aeroplanes these nine quantities can be expressed for the separate planes in terms of $f'(\alpha)$ and $\phi'(\alpha)$, where $f(\alpha)$ and $\phi(\alpha)$ are functions determining the resultant thrust, and the position of the centre of pressure when the direction of the relative motion of the air makes an angle α with the plane. These functions have been tabulated for certain different forms of surfaces, but further data are greatly needed.

4. *The longitudinal stability of the gliders is thus seen to be capable of mathematical investigation*, and it is of paramount importance that the present methods should be practically applied to any aerial machines that may be designed or constructed before any actual glides are attempted.

5. The methods of calculation are exemplified by numerical determinations of the criterion of stability in the cases of a single plane lamina and a pair of planes one behind the other. Most of the calculations have been performed for an angle of gliding of 10° with the horizon, and it has been necessary to assume arbitrary values for the moment of inertia of the lamina.

6. The condition that any steady linear motion may be stable in all these cases assumes the form $V^2 > ka$, where a is a constant depending on the linear dimensions of the glider, and k is a constant depending on its shape, the angle of gliding and the law of aerial resistance.

7. For a pair of narrow slats, in which the variations in the positions of the centres of pressure of each are neglected, certain coefficients of stability vanish. If the planes are square so that the displacements of the centres of pressure are not neglected, the system is less stable than a single plane of breadth equal to one of the squares.

8. By inclining the planes at a small angle to each other the stability

is much increased. On the other hand, if they are made to slope away from each other the glider becomes unstable.

9. Two square planes of equal size placed one behind the other at a small angle are less stable in the examples considered than a square equal in size to one of the two, but more stable than a single square whose side is equal to the total length of the glider formed by the pair.

10. A pair of unequal squares of which the smaller forms a rudder are more stable, in the examples considered, when gliding with the rudder behind than with the rudder in front.

11. In general, the stability is increased by making the moment of inertia as small as possible.

"Cultural Experiments with 'Biologic Forms' of the *Erysiphaceæ*."

By Ernest S. Salmon, F.L.S. Communicated by Professor H. Marshall Ward, F.R.S. Received December 2, 1903,—Read February 4, 1904.

(Abstract.)

In the introductory remarks the author points out that through specialisation of parasitism "biologic forms" have been evolved in the *Erysiphaceæ* which, both in their conidial (asexual) stage and ascigerous (sexual) stage, show specialised and restricted powers of infection. The powers of infection, characteristic of each "biologic form," are under normal conditions sharply defined and fixed, and hitherto the result of the experiments of numerous investigators—both in regard to the present group of fungi and to the *Uredineæ*, where the same specialisation of parasitism occurs—has been the accumulation of evidence tending to emphasise the immutability of "biologic forms."

The second part of the paper gives the result of cultural experiments with "biologic forms" of *Erysiphe Graminis* DC., carried out during the past summer in the Cambridge University Botanical Laboratory. It has been found that *under certain methods of culture, in which the vitality of the host-leaf is interfered with, the restricted powers of infection, characteristic of "biologic forms," break down.*

In the first method of culture adopted, the leaf, which was either attached to a growing plant, or removed and placed in a damp chamber, was injured by the removal of a minute piece of leaf-tissue. In this operation the epidermal cells on one surface, and all or most of the mesophyll tissue, were removed at the cut place, but the epidermal cells on the other surface (opposite the cut) were left uninjured. Conidia were sown on the cuticular surface of the uninjured epidermal cells over the cut. In a few experiments the conidia were